In a nutshell: Approximating solutions to systems of 1st-order initial value problems

Given a system of n coupled initial-value problems (IVPs)

$$\begin{aligned} y_1^{(1)}(t) &= f_1\left(t, y_1(t), y_2(t), \ldots, y_n(t)\right) \\ y_1(t_0) &= y_{1,0} \\ y_2^{(1)}(t) &= f_2\left(t, y_1(t), y_2(t), \ldots, y_n(t)\right) \\ y_2(t_0) &= y_{2,0} \\ &\vdots \\ y_n^{(1)}(t) &= f_n\left(t, y_1(t), y_2(t), \ldots, y_n(t)\right) \\ y_n(t_0) &= y_{n,0} \end{aligned}$$

write this as

$$\mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix}, \quad \mathbf{y}^{(1)}(t) = \begin{pmatrix} y_1^{(1)}(t) \\ y_2^{(1)}(t) \\ \vdots \\ y_n^{(1)}(t) \end{pmatrix} = \mathbf{f}\left(t, \mathbf{y}(t)\right) = \begin{pmatrix} f_1\left(t, y_1(t), y_2(t), \ldots, y_n(t)\right) \\ f_2\left(t, y_1(t), y_2(t), \ldots, y_n(t)\right) \\ \vdots \\ f_n\left(t, y_1(t), y_2(t), \ldots, y_n(t)\right) \end{pmatrix}, \text{ and } \mathbf{y}(t_0) = \begin{pmatrix} y_{1,0} \\ y_{2,0} \\ \vdots \\ y_{n,0} \end{pmatrix} = \mathbf{y}_0 .$$

Now, $\mathbf{y}$ is a vector-valued function of a real variable t, and $\mathbf{y}^{(1)}(t) = \mathbf{f}\left(t, \mathbf{y}(t)\right)$ is a vector equation with an initial condition being a vector. Now, for each of the techniques we have seen to date, they can be modified as follows:

1. Euler: $\mathbf{y}_{k+1} \leftarrow \mathbf{y}_k + h\mathbf{f}\left(t_k, \mathbf{y}_k\right)$.
2. Heun: $\mathbf{s}_0 \leftarrow \mathbf{f}\left(t_k, \mathbf{y}_k\right)$, $\mathbf{s}_1 \leftarrow \mathbf{f}\left(t_k + h, \mathbf{y}_k + h\mathbf{s}_0\right)$, $\mathbf{y}_{k+1} \leftarrow \mathbf{y}_k + h\dfrac{\mathbf{s}_0 + \mathbf{s}_1}{2}$.
3. 4th-order Runge-Kutta:
   $\mathbf{s}_0 \leftarrow \mathbf{f}\left(t_k, \mathbf{y}_k\right), \mathbf{s}_1 \leftarrow \mathbf{f}\left(t_k + ½h, \mathbf{y}_k + ½h\mathbf{s}_0\right), \mathbf{s}_2 \leftarrow \mathbf{f}\left(t_k + ½h, \mathbf{y}_k + ½h\mathbf{s}_1\right), \mathbf{s}_3 \leftarrow \mathbf{f}\left(t_k + h, \mathbf{y}_k + h\mathbf{s}_2\right)$

   and $\mathbf{y}_{k+1} \leftarrow \mathbf{y}_k + h\dfrac{\mathbf{s}_0 + 2\mathbf{s}_1 + 2\mathbf{s}_2 + \mathbf{s}_3}{6}$
4. Our adaptive Euler-Heun method is similarly modified where we now calculate approximations $\mathbf{y}$ and $\mathbf{z}$ and then set $a \leftarrow \dfrac{h\varepsilon_{\text{abs}}}{2\left\|\mathbf{y} - \mathbf{z}\right\|_2}$ where $\left\|\mathbf{y} - \mathbf{z}\right\|_2$ is the 2-norm or Euclidean norm calculating the magnitude of the difference of the two approximating vectors.
5. The adaptive Dormand-Prince method is similarly modified where we now calculate approximations $\mathbf{y}$ and $\mathbf{z}$ and then set $a \leftarrow \sqrt[4]{\dfrac{h\varepsilon_{\text{abs}}}{2\left\|\mathbf{y} - \mathbf{z}\right\|_2}}$.